Hash functions from superspecial genus-2 curves using Richelot isogenies

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NutMiC 2019, Paris

June 24, 2019

Background

- 2006: hash functions based on supersingular elliptic curves (Charles, Goren, Lauter)
- 2011: key exchange protocol based on supersingular elliptic curves called SIDH (Jao, De Feo)

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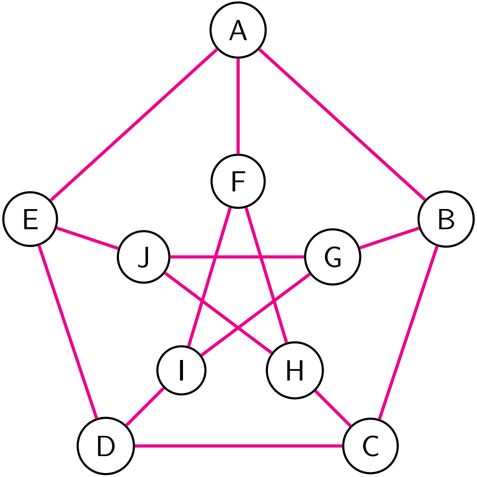
- 2011: key exchange protocol based on supersingular elliptic curves called SIDH (Jao, De Feo)
- 2018: hash function based on supersingular genus-2 curves (Takashima)
- 2019: collisions in genus-2 hash, create genus-2 SIDH (Flynn, Ti)
- 2019: we fix collisions and smooth out a bunch of technicalities

Hash functions from expander graph

Input: 110



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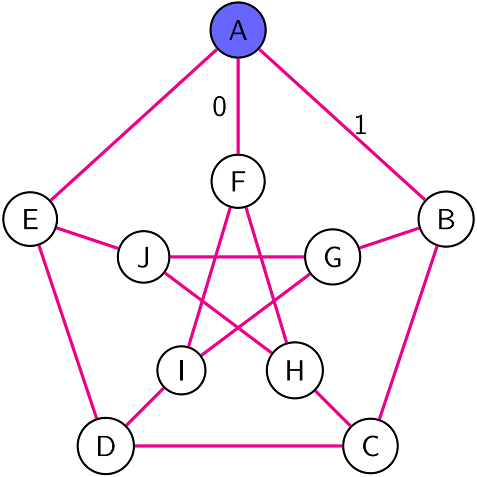

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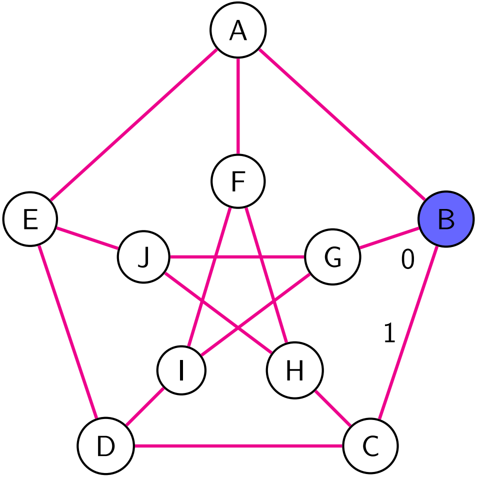

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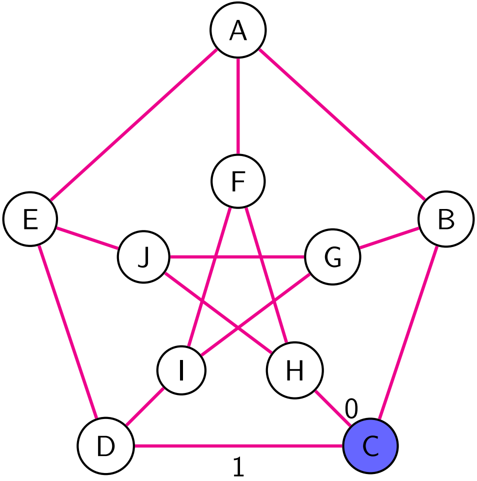
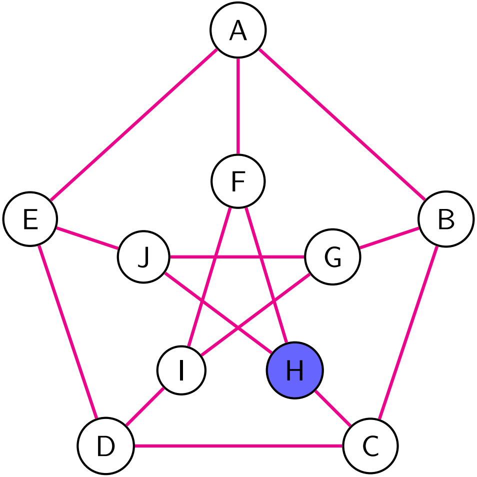


Hash functions from expander graph

Input: 110; Output: H



Supersingular ℓ-isogeny graph over $\mathbb{F}_{p^2}$

Construct the graph $G(p, \ell)$ as follows:

- Vertices: all supersingular elliptic curves over $\mathbb{F}_{p^2}$ up to $\cong$
- Edges: all ℓ-isogenies between them

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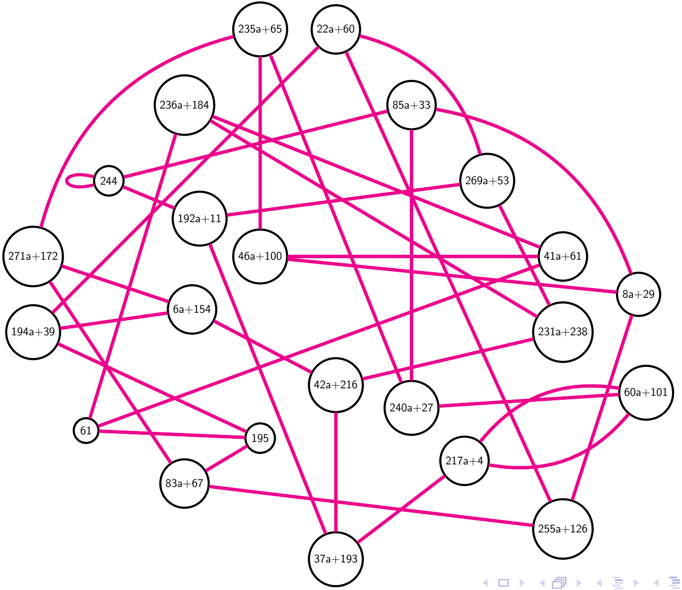
Some properties:

- Amount of vertices $\sim p/12$
- Good expander graph
- Every node has $\ell + 1$ edges

$G(277,2)$ with $\mathbb{F}_{277^2} \cong \mathbb{F}_{277}(a) \cong \mathbb{F}_{277}[x]/(x^2+274x+5)$



Security

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Given two supersingular elliptic curves E and E' defined over $\mathbb{F}_{p^2}$, find an ℓ^k-isogeny between them.

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Given any supersingular elliptic curve E defined over $\mathbb{F}_{p^2}$, find a curve E' and two distinct isogenies of degree ℓ^k and $\ell^{k'}$ between them.

General idea

2-isogenies between supersingular elliptic curves

$$\downarrow$$

(2,2)-isogenies between principally polarized superspecial abelian surfaces

Elliptic curves

Definition

An elliptic curve, say E, over a field K of odd characteristic, is an algebraic curve defined by an equation of the form

$$E : y^2 = f(x),$$

where $f(x)$ is a squarefree polynomial in $K[x]$ of degree 3 or 4.

Genus two curves

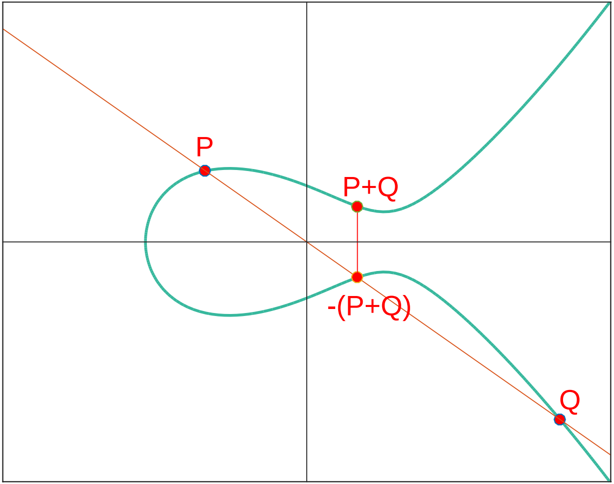
Definition

A hyperelliptic curve of genus two, say C, over a field K of odd characteristic, is an algebraic curve defined by an equation of the form

$$C : y^2 = f(x),$$

where $f(x)$ is a squarefree polynomial in $K[x]$ of degree 5 or 6.

Elliptic curves group law



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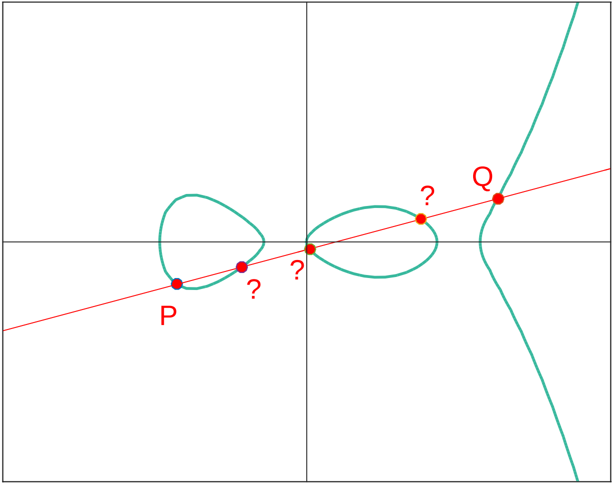

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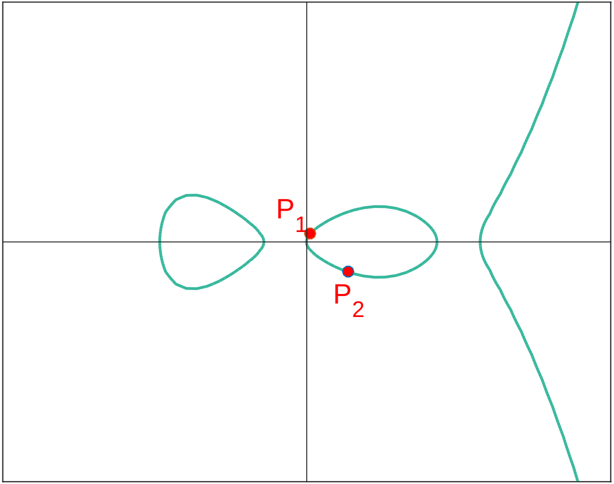

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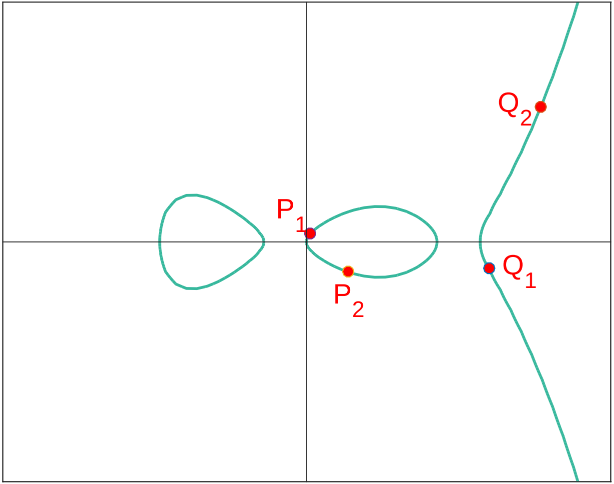

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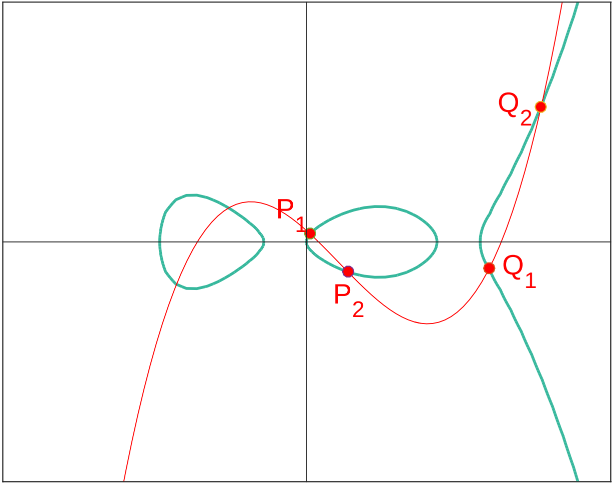

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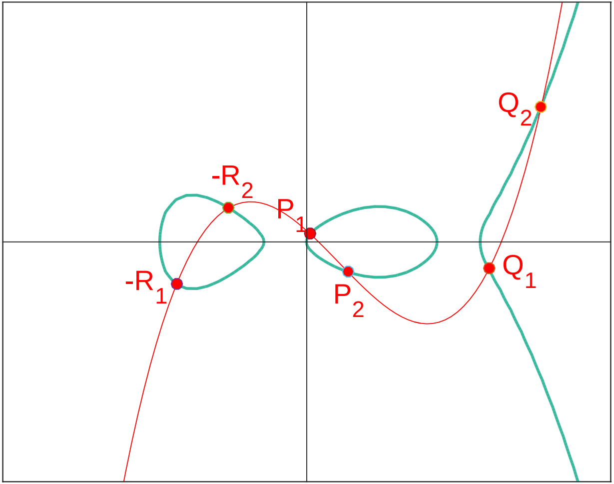

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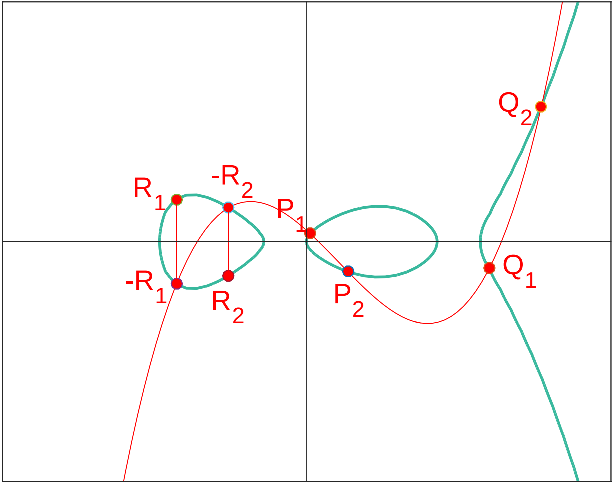

Abelian surfaces

Definition

An **abelian surface** is a two-dimensional projective algebraic variety that is also an algebraic group.

Always isomorphic to one of the following:

- jacobian of a (hyperelliptic) genus-2 curve
- product of two elliptic curves

Principal polarization

Definition

A **principal polarization** is an isomorphism λ from an abelian variety A to its dual, which is of the form

$$\begin{aligned} \lambda_{\mathcal{L}} : A(\bar{k}) &\rightarrow \mathrm{Pic}(A) \\ a &\mapsto t_a^* \mathcal{L} \otimes \mathcal{L}^{-1}, \end{aligned}$$

for some ample sheaf $\mathcal{L}$ on $A(\bar{k})$.

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Read: we have equations!

- $y^2 = a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$
- $(y^2 = x^3 + b_1x + b_0) \times (y^2 = x^3 + c_1x + c_0)$

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- or the Newton polygon is a straight line segment with slope $1/2$,
- or the dual of Frobenius is purely inseparable,
- or the Hasse invariant is 0,
- $\ldots$

Superspecial genus two curves

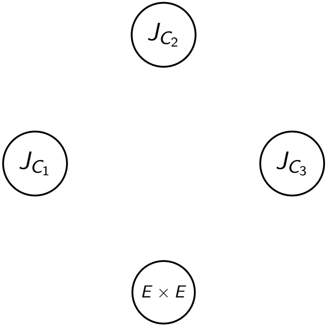
Definition

A p.p. abelian surface defined over a field with characteristic p is **superspecial** if the Hasse invariant is zero.

Why?

- Finite amount $\sim p^3/2880$
- All defined over $\mathbb{F}_{p^2}$

Superspecial abelian surfaces over $\mathbb{F}_{13^2}$



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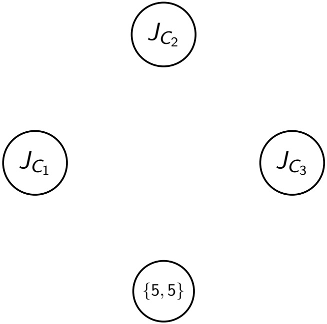

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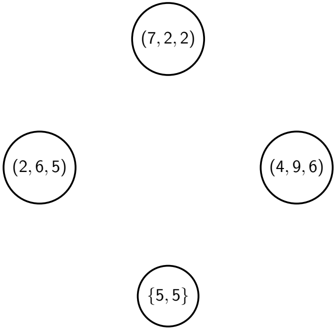


$(2,2)$-isogenies

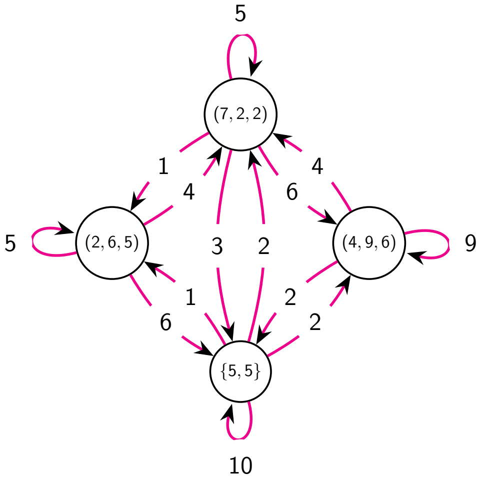
Definition

A $(2,2)$-**isogeny** ϕ is an isogeny such that $\ker\phi \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $\ker\phi$ is maximal isotropic with regards to the 2-Weil pairing.

Remark: there are 15 of these $(2,2)$-isogenies for every A, and at least 9 are to the same type of abelian surface, so

$$J_C \to J_{C'} \text{ or } E_1 \times E_2 \to E_1' \times E_2'$$

Superspecial p.p. abelian surface $(2,2)$-isogeny graph over $\mathbb{F}_{13^2}$



Superspecial p.p. abelian surface $(2,2)$-isogeny graph over $\mathbb{F}_{p^2}$

Isogeny graph $\mathcal{G}_p$:

- Vertices: all p.p. superspecial abelian surfaces over $\mathbb{F}_{p^2}$ up to isomorphism
  - genus-2 curves: absolute Igusa invariants $(j_1, j_2, j_3) \in \mathbb{F}_{p^2}^3$
  - products of elliptic curves: j-invariants $\{j_1, j_2\} \subset \mathbb{F}_{p^2}$
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Intuitively:

- Interior of $\mathcal{G}_p$: $\sim p^3/2880$ genus-2 curves
- Boundary of $\mathcal{G}_p$: $\sim p^2/288$ products of elliptic curves

Restrict to jacobians of genus-2 curves

Ignore products of elliptic curves:

- $\mathcal{O}(1/p)$ chance of encountering
- formulas are less efficient
- what would output be? $\{j_1, j_2\}$ vs (j_1, j_2, j_3)

Richelot isogenies

$$C_0 : y^2 = \underbrace{(x-\alpha_1)(x-\alpha_2)}_{G_1}\underbrace{(x-\alpha_3)(x-\alpha_4)}_{G_2}\underbrace{(x-\alpha_5)(x-\alpha_6)}_{G_3}$$

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Take $\phi_1 : J_{C_0} \to J_{C_1}$ the $(2,2)$-isogeny with kernel

$$\{0, [(\alpha_1, 0) - (\alpha_2, 0)], [(\alpha_3, 0) - (\alpha_4, 0)], [(\alpha_5, 0) - (\alpha_6, 0)]\}$$

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$$\rightsquigarrow C_1 : y^2 = \delta^{-1}\underbrace{(G_2'G_3 - G_2G_3')}_{H_1}\underbrace{(G_3'G_1 - G_3G_1')}_{H_2}\underbrace{(G_1'G_2 - G_1G_2')}_{H_3}$$

Avoiding dual isogeny

Continuing with $y^2 = H_1 H_2 H_3$ gives the dual isogeny $\hat{\phi}_1$ and the composition is a $(2, 2, 2, 2)$-isogeny:

$$\begin{tikzcd} A_0 \arrow[r, bend left, "\phi_1"] & A_1 \arrow[l, bend left, "\hat{\phi}_1"] \end{tikzcd}$$

Avoiding small cycles

Continuing with one factor fixed, e.g. $y^2 = H_1\tilde{H}_2\tilde{H}_3$, gives a $(2,2)$-isogeny ϕ_2, with a composed $(4,2,2)$-isogeny:

$$A_0 \xrightarrow{\phi_1} A_1 \xrightarrow{\phi_2} A_2$$

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$$\begin{tikzcd}
 & A_1' \arrow[dr, "\phi_2'"] & \\
A_0 \arrow[ur, "\phi_1'"] \arrow[r, "\phi_1"] \arrow[dr, "\phi_1''"'] & A_1 \arrow[r, "\phi_2"] & A_2 \\
 & A_1'' \arrow[ur, "\phi_2''"'] &
\end{tikzcd}$$

Good isogeny extensions

Write $H_1 = L_1 L_2$, $H_2 = L_3 L_4$, $H_3 = L_5 L_6$ then the good extensions of ϕ_1 are determined by the quadratic factors

$$\begin{array}{ll}
(L_1 L_3, L_2 L_5, L_4 L_6), & (L_1 L_3, L_2 L_6, L_4 L_5), \\
(L_1 L_4, L_2 L_5, L_3 L_6), & (L_1 L_4, L_2 L_6, L_3 L_5), \\
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(L_1 L_6, L_2 L_3, L_4 L_5), & (L_1 L_6, L_2 L_4, L_3 L_5).
\end{array}$$

Composing gives a $(4, 4)$-isogeny.

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Given two superspecial genus-2 curves C_1 and C_2 defined over $\mathbb{F}_{p^2}$, find a $(2^k, 2^k)$-isogeny between their jacobians.

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1. *a curve C_2 and a $(2^k, 2^k)$-isogeny $J_{C_1} \to J_{C_2}$,*
2. *a curve C_2' and a $(2^{k'}, 2^{k'})$-isogeny $J_{C_1} \to J_{C_2'}$,*

such that C_2 and C_2' are $\overline{\mathbb{F}}_p$-isomorphic.

Concluding remarks

Advantages:

- Processing 3 bits at once, with possible parallelization of 3 square root extractions
- Elliptic curves graph size $\mathcal{O}(p)$
  Genus-2 curves graph size $\mathcal{O}(p^3)$

  $\Rightarrow$ same security in smaller fields, e.g. $p \approx 2^{86}$ vs $p \approx 2^{256}$

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Future research:

- Practical genus-2 SIDH key exchange?
- Expander properties of $\mathcal{G}_p$?