Delegating a Product of Group Exponentiations with Application to Signature Schemes

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Introduction and Motivation

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- Preliminary solutions to this problem considered mostly honest servers or multiple servers of which at least one is honest. In the case of a **single**, possibly **malicious server**, this problem has remained open since a formal cryptographic model was introduced [HL'05].
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- In [DKKS'17] we solved this problem for a large class of cyclic groups.
- In this paper, we show how to delegate a product of (fixed-base) exponentiations, in a large class of cyclic groups.

Problem History and Related Work

- The **problem** of outsourcing exponentiation to a single malicious server was formally defined and posed in [HL'05] (and studied even earlier), where they gave protocols in the case of **2 servers** of which at most one was malicious, and to **1 server, who was honest on almost all inputs**.

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- Several other solutions consider
  - delegation of **general functions**, or
  - delegation of **linear algebra functions**.
- Closest result is our previous [DKKS'17] paper where we solve the above open problem for the delegation of a **single** fixed-base exponentiation in a large class of cyclic groups.

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- Consider natural question, motivated by [HL'05] and encryption/signature literature, of whether we can efficiently delegate the **product of multiple exponentiations**.

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- We show **a protocol** for the delegation to **a single (malicious) server** of a product of fixed-base exponentiations in a large class of cyclic groups
  - This improves the client's number of group multiplications by a factor of about σ with respect to non-delegated computation and a factor of about m with respect to direct use of [DKKS'17].

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- We use this result to delegate the **first cryptographic schemes**: the well-known digital signature schemes by El-Gamal, Schnorr and Okamoto. Previously, only primitive operations were delegated.

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- We use this result to delegate the **first cryptographic schemes**: the well-known digital signature schemes by El-Gamal, Schnorr and Okamoto. Previously, only primitive operations were delegated.
- In the process, we formally define delegation of digital signature schemes and prove a **conversion theorem** showing that a non-delegated to delegated signature scheme conversion using a suitable delegation protocol for a desired primitive operation

Delegation Protocols: Participant and Interaction Model

Offline phase

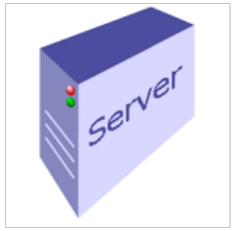
run by Client Deployer or Client, resulting in data stored on Client's device

Online phase

C(x, offline data)

Client's message

Server's message



S

Return: y=F(x)

Delegation Protocols: Algorithm Syntax and Requirements

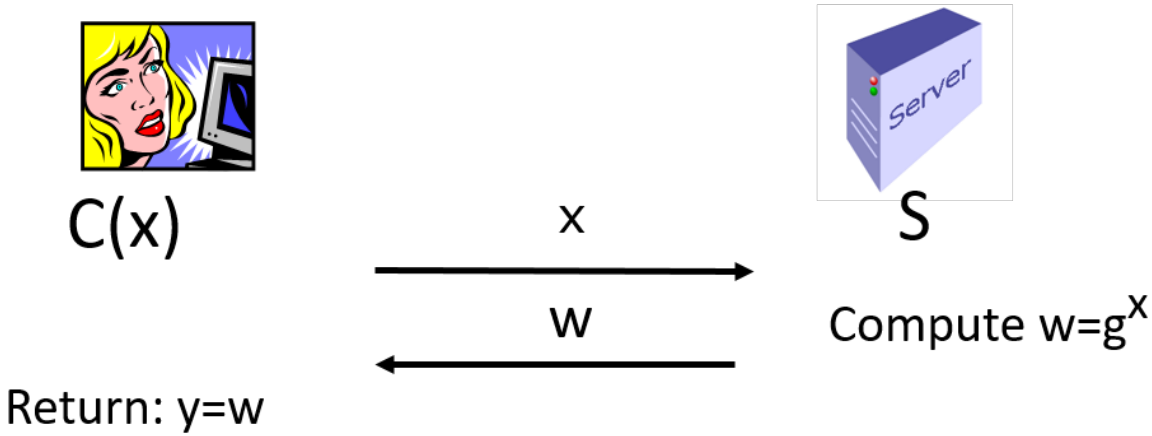
- Model based on [DKKS'17], in turn based on [HL'05] and [GGP'10]
- **Input to client C and server S:** σ (computational parameter), λ statistical parameter, $desc(F)$ (description of delegated function F)
- **Input to C:** x
- **C's output:** $F(x)$, or $\perp$ (failure symbol)
- **Requirements:**
  - **Correctness:** If both parties are honest, C obtains some output at the end of the protocol, and this output should be, with high probability, $= F(x)$, the value obtained by evaluating function F on C's input.
  - **Privacy:** minimal or no information about x should be revealed to S
  - **Security:** a malicious S should not be able, except with very small probability, to convince C to output a non-failure result $\neq F(x)$
  - **Efficiency:** C's computation time should be much smaller than computing $F(x)$ without delegating computation; also runtime of S, runtime in offline mode, round and communication complexity should be kept small.
- Offline computation (by C or others) is allowed before C is given x

Our Delegated Function and Efficiency Requirements

- 1^σ is a computational security parameter (e.g., $\sigma = 2048$)
- 1^λ is a statistical security parameter (e.g., $\lambda = 128$)
- $F_{g_1,\ldots,g_m}(x_1,\ldots,x_m) = y$ where $y = \prod_{i=1}^{m} g_i^{x_i}$ such that
  - $g_1, \ldots, g_m$ are generators of cyclic group $(G, *)$
  - $|g_i| = \sigma$, for $i = 1, \ldots, m$
- Note that $\text{feExp}_{n,e}$ can be evaluated in
  - $\leq 2m\sigma + m - 1$ multiplications, using square-and-multiply algorithm
  - $\leq \frac{2m\sigma}{\log(m\sigma)} + \sigma +$ o-factor multiplications, using improved 'multi-exponentiation' algorithms, including Pippenger's 1976 algorithm
- Main efficiency goals for a protocol (C, S) for a delegated computation of function $F_{g_1,\ldots,g_m}$:
  - C's group multiplications $\ll m\sigma$ and
  - t_S is not significantly larger than t_F.

Towards delegating 1 exponentiation: simplest attempt

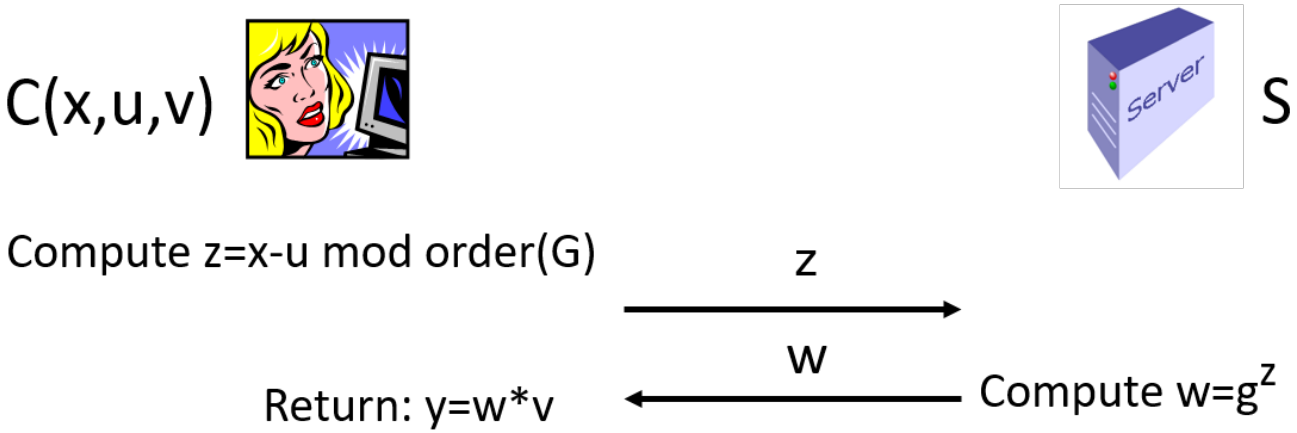
Let G be a cyclic group with generator g known to both C and S



- *Correctness* holds: C obtains $w = g^x$ (here, S is honest)
- *Privacy* is not satisfied: S learns x
- *Security* is not satisfied: C cannot trust its result y (here, S may not be honest)
  - Note: C is computationally weak and cannot compute g^x
  - this also rules out using proofs of knowledge of a dlog exponent

Towards delegating 1 exponentiation: input masking

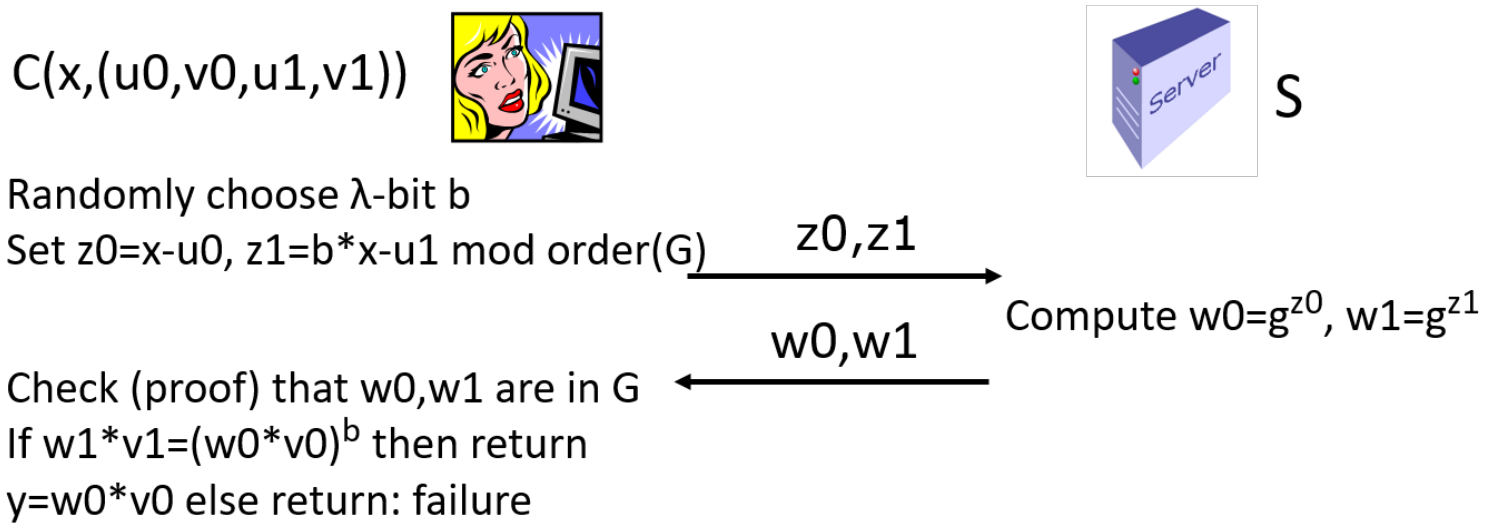
Let G be a cyclic group with generator g known to both C and S. Someone (Deployer or Client) precomputes (u,v), where u is random in {1,..,order(G)} and $v=g^u$, and stores (u,v) on Client.



- *Correctness* holds: C obtains $w = g^x$ (here, S is honest)
- *Privacy* is satisfied: no S can learn any information about x
  - Input masking gives us privacy
- *Security* is still not satisfied: C cannot trust its result y (here, S may not be honest)
- Assuming order(G) is known to C

Protocol from [DKKS17] to delegate 1 exponentiation

Deployer or Client precomputes (u[j],v[j]), where u[j] is random in {1,..,order(G)} and $v[j]=g^{u[j]}$, and stores (u[j],v[j]), j=0,1, on Client

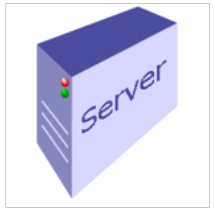


- *Ingredients*: input masking, small-coefficient linear test, efficient group membership test
- Satisfies *Correctness*, *Privacy*, *Security* and *Efficiency*; C performs $\leq 2\lambda + 3$ multiplications (e.g., 259); compare with $\leq 2\sigma$ (e.g., 4096)
- *Group assumptions*: cyclic, g known, efficient group membership test

Using [DKKS17] protocol for a product of exponentiations

Deployer or Client precomputes (u[j],v[j]), where u[j] is random in {1,..,order(G)} and $v[j]=g^{u[j]}$, and stores (u[j],v[j]), j=0,1, on Client

C(x[1],...,x[m]) 



S

Randomly choose λ-bit b[1],...,b[m]
Set z[0i]=x+u[0i], z[1i]=b[i]*x+u[1i]

z[0i],z[1i] →

← w[0i],w[1i]

Compute $w0i=g^{z0i}$, $w1i=g^{z1i}$

Check that w[0i],w[1i] are in G
If $w[1i]/v[1i]=(w[0i]/v[0i])^{b[i]}$ then compute
y[i]=w[01]/v[01] and return y=y[1]*...*y[m]
else return: failure

- Essentially a parallel repetition of [DKKS17]
- C does $\leq 2\lambda m + 4m - 1 << \sigma m$ multiplications (linear in λm)

Ideas behind our new protocol

- Peeliminary discussion:
  - In previous protocol gap between S′s runtime and Cs runtime becomes less practically relevant as m gets large; but products of exponentiations are frequent in crypto protocols. Can we do better?
  - Can C verify m simultaneous exponentiations faster than independently verifying all m of them? No, or not much faster
  - Does C need to compute m simultaneous exponentiations? No, only a product of m exponentiations
- Ideas behind our new protocol:
  - Input masking on each of the m exponentiations
  - Small-coefficient linear test is now run on the product of exponentiations instead of on the single exponentiation; can use a single test (instead of m)
  - Let S and offline deployer compute products of exponentiations (instead of C)

Our new protocol for a product of exponentiations

Deployer or Client precomputes (u[ij],v[ij]), where u[ij] is random in {1,..,order(G)} and v[j]= $\prod g^{u[ij]}$, and stores (u[1j],...,u[mj],v[j]), j=0,1, on Client

C(x[1],...,x[m]) 

S

Randomly choose λ-bit b
Set z[0i]=x+u[0i], z[1i]=b*x+u[1i]

z[0i],z[1i] →

← w0,w1

Check that w0,w1 are in G
If w1/v1=(w0/v0)^b then return y=w0/v0 else return: failure

Compute
w0= $\prod g^{z[0i]}$
w1= $\prod g^{z[1i]}$

- *Ingredients*: masking m inputs, single small-coefficient linear test, only 2 efficient group membership tests
- Provable correctness, privacy, $2^{-\lambda} - security$, efficiency
- C does $\leq 2\lambda + 3m << \sigma m$ multiplications (linear in $\lambda + m$)

Analysis of Client Runtime

Table 1. Comparison of number of C's online multiplications in $\mathbb{Z}_p^*$ where $p = 2q + 1$

	m	$F_{g_1,\ldots,g_m,q}$ No delegation		$F_{g_1,\ldots,g_m,q}$ With delegation	
		nPoExp	*fPoExp*	*nDelPoExp*	Our result
$\sigma = 2048$ $\lambda = 128$	2	8,193	6,144	520	261
	5	20,484	12,288	1,300	264
	10	40,969	22,528	2,600	269
	50	204,849	104,448	13,000	309
	100	409,699	206,848	26,000	359
	1000	4,096,999	2,050,048	260,000	1,259
	Arbitrary m	$4,097m - 1$	$2,048(m + 1)$	$260m$	$259 + m$
Arbitrary m, σ, λ		$2m\sigma + m - 1$	$m\sigma + \sigma$	$2m\lambda + 4m$	$2\lambda + m + 3$

- *Main Takeaways*:
  - Improvement with respect to undelegated computation is a multiplicative factor of about σ
  - Improvement with respect to parallel use of [DKKS17] delegation of single exponentiation is a multiplicative factor of about m

Applications: Delegation to Signature Schemes

1. Main result: we show efficient, private and secure delegation schemes for well-known (i.e., ElGamal, Schnorr and Okamoto's) signature schemes using the delegation of a product of (fixed-base) exponentiation for cyclic groups
2. We augment the unforgeability definition of (non-delegated) signature scheme so to additionally take into account eavesdropping and oracle query attacks in the delegated model.
3. We present a general result that shows how to convert signature schemes in the non-delegated model into signature schemes in the delegated model by using a suitable delegation protocol.
4. Finally, we show delegated ElGamal, Schnorr and Okamoto's signature schemes.

Non-Delegated Signature Schemes: definitions review

- 3 Algorithms:
  - Key generation returns Alices pk and sk
  - With Sign, Alice can compute a signature sig for message m using pk,sk,
  - With Verify, Bob can verify (m,sig) using Alices pk
- 2 requirements:
  - Correctness: Bob can verify Alices (mes,sig) pair based on Alices pk
  - Unforgeability: Even after polynomial adaptive queries to the Sign(pk,.) oracle, no efficient adversary can forge a signature for a new message, unless with negligible probability

Delegated Signature Schemes: definitions

- *Syntax* of delegated signature schemes:
  - signer and verifier are oracle algorithms that may use, as an oracle, the delegation server
  - both the signer and the verifier can act as client in a (C,S) instance
- *Correctness* is a direct extension of the non-delegated case
- For *unforgeability*, we replace the adversary A's oracle $Sign(pk, sk, \cdot)$ with two oracles:
  - an augmented oracle $dSign(pk; sk; \cdot)$ which, on input message m, returns a signature *sig* and the transcript of any query/answer interaction with the server oracle S performed by Sign during generation of *sig*; this models *chosen-message* and *eavesdropping attacks*;
  - the server oracle $S(desc(F); \cdot)$, which, on input C's query message, returns S's response to this message in an execution of protocol (C,S); this models *collusion with a client or a server*.

Delegated Signatures: results

- (Informal) General Theorem: Given a secure signature scheme where signer and/or verifier use F and and a delegation protocol for F we can construct a secure delegated signature scheme where signer and/or verifier delegates F
  - Note: Privacy of delegation protocol has to be based on a simulation-based definition, which we achieve
  - Consequence: Can replace any computation by signers and verifiers with its delegation and keep security
  - We replace the most expensive step in the verifier's computation of a product of 2 or 3 exponentiations in the well-known signature schemes by ElGamal, Schnoor and Okamoto, using our new delegation protocol
    - The number of online group multiplications by the verifier is reduced by a factor between 20 and 40, and remains less than 260 in the worst case

Conclusions

- We solved the problem of delegating the computation of a product of group exponentiations to a single, possibly malicious, server, in a large class of cyclic groups; specifically, cyclic groups whose multiplication and inverse operations can be efficiently computed, and which admit an efficiently verifiable protocol to prove that an element is in the group. The considered class of cyclic groups includes groups often discussed in cryptography literature, such as prime-order subgroups in Z_p and elliptic curve groups.
  - Future work: working on more groups, including post-quantum ones; p-groups, etc.
- As an application, we showed the first delegation of an entire cryptographic scheme. Previous research only delegated a single operation of a scheme's algorithm.
  - Future work: other, more general, schemes?
- Open problem: This paper, and our previous 2 papers are on delegation of fixed-base exponentiation. Private, secure and efficient delegation of (even single) variable-base exponentiation is open.

Thank You!

Questions?

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