New Zémor-Tillich Type Hash Functions Over $\mathrm{GL}_2(\mathbb{F}_{p^n})$

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What is a Cayley Hash?

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Associated Cayley hash

Given a group G and $g_1, g_2 \in G$, the **associated [Cayley] hash** H is the map defined for any message $m = m_1 \dots m_k \in \{0,1\}^*$ by $H(m) = H(m_1) \cdots H(m_k) \in G$ where $H(0) = g_1$ and $H(1) = g_2$.

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Small modifications property

Given any collision $H(m) = H(m')$, $\min\{|m|, |m'|\} \geq n$.

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Some examples

- Zémor's original suggestion was to use $g_1 = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $g_2 = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ in $\mathrm{SL}_2(\mathbb{F}_p)$ for p a large prime
- Cayley hashes from expander graphs
- Bromberg et. al. suggested using pairs of the form $g_1 = \left[\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right]$ and $g_2 = \left[\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right]$ in $\mathrm{SL}_2(\mathbb{F}_p)$

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Attacks

- small order attacks (Charnes and Piepryzk, Steinwandt et. al.)
- Geiselmann's embedding attack
- Grassl et. al's palindrome attack

Our contribution

Our hash function construction: *Let* $A, B \in \mathrm{M}_{2\times 2}\left(\mathbb{F}_p[x]\right)$ *and set* $\mathcal{D}$ *to be* $\left\{M \in \mathrm{M}_{2\times 2}\left(\mathbb{F}_p[x]\right) \mid r_n \nmid \det(M)\right\}$. *Define the projection map*

$$\pi_{r_n} : \mathcal{D} \to \mathrm{GL}_2\left(\mathbb{F}_q\right)$$

to be the map taking entries of a matrix to their projection in $\mathbb{F}_q$ *under the quotient by* $\langle r_n \rangle$. *We then construct a hash function* H *by taking the associated hash for* $g_1 = \pi_{r_n}(A)$ *and* $g_2 = \pi_{r_n}(B)$ *and* $G = \mathrm{GL}_2\left(\mathbb{F}_{p^n}\right)$.

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Our idea:

Use freeness to retain the small modifications property.

The field of **formal Laurent series** over $\mathbb{F}_p$

The elements of $\mathbb{F}_p((x))$ are series of the form

$$g(x) = \sum_{k=m}^{\infty} g_k x^k$$

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- $\mathrm{GL}_2\left(\mathbb{F}_p((x))\right)$
- $\mathbb{P}^1$

Free Generators Theorem

Free Generators Theorem (T. 2018)

Let p be a prime and let $d \in \mathbb{N}_0$. Suppose there exist $a, b, c, \tilde{a}, \tilde{b} \in \mathbb{F}_p((x))$, $f, \tilde{f} \in \mathbb{F}_p((x))^\times$, such that Ξ_1, Ξ_2 and Ξ_3 hold (see next slide). Then the matrices

$$A = \begin{bmatrix} ab - cf & a(f-1) \\ cb(1-f) & abf - c \end{bmatrix} \text{ and } B = \begin{bmatrix} \tilde{b} - \tilde{a}\tilde{f} & \tilde{f} - 1 \\ \tilde{a}\tilde{b}(1-\tilde{f}) & \tilde{b}\tilde{f} - \tilde{a} \end{bmatrix} \tag{1}$$

generate a free group in $\mathrm{PGL}_2\big(\mathbb{F}_p((x))\big)$. In particular, any inverse images of A, B in $\mathrm{GL}_2\big(\mathbb{F}_p((x))\big)$ also generate a free group.

Conditions of the Free Generators Theorem

Ξ_1 : $d([u],[v]) > \frac{1}{p^{d+1}}$ for each pair of $[u],[v]$ in $\left\{[a:c],[1:b],[1:\tilde{a}],[1:\tilde{b}]\right\}$

Ξ_2 : $\min\{|f|,|f^{-1}|\} \leq \frac{1}{p^{2d+1}}$, and $\min\{|\tilde{f}|,|\tilde{f}^{-1}| \leq \frac{1}{p^{2d+1}}\}$

Ξ_3 : There exists $[z] \in \mathbb{P}^1$ such that $d([z],[u]) > \frac{1}{p^{d+1}}$ for each $[u]$ in $\left\{[a:c],[1:b],[1:\tilde{a}],[1:\tilde{b}]\right\}$.

Remark

We can find infinitely many parameters satisfying our theorem for all $d \geq 0$ when p is odd, and all $d > 0$ when $p = 2$.

Some constructions using the Free Generators Theorem

Table: The matrices A and B produced using the Free Generators Theorem for $p > 2$, $d = 0$, $a = 0$, $c = 1$, $f, \tilde{f} \in x\mathbb{F}_p[x]$, and given choices of b, $\tilde{a}$, and $\tilde{b}$.

$\{A, B\}$	A	B	b	$\tilde{a}$	$\tilde{b}$
$G_1(f, \tilde{f})$	$\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \tilde{f}+1 & 1-\tilde{f} \\ 1-\tilde{f} & \tilde{f}+1 \end{pmatrix}$	0	1	-1
$G_2(f, \tilde{f})$	$\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \tilde{f}+1 & \tilde{f}-1 \\ \tilde{f}-1 & \tilde{f}+1 \end{pmatrix}$	0	-1	1
$G_3(f, \tilde{f})$	$\begin{pmatrix} f & 0 \\ f-1 & 1 \end{pmatrix}$	$\begin{pmatrix} \tilde{f} & \tilde{f}-1 \\ 0 & 1 \end{pmatrix}$	1	-1	0
$G_4(f, \tilde{f})$	$\begin{pmatrix} f & 0 \\ f-1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1-\tilde{f} \\ 0 & \tilde{f} \end{pmatrix}$	1	0	-1
$G_5(f, \tilde{f})$	$\begin{pmatrix} f & 0 \\ 1-f & 1 \end{pmatrix}$	$\begin{pmatrix} \tilde{f} & 1-\tilde{f} \\ 0 & 1 \end{pmatrix}$	-1	1	0
$G_6(f, \tilde{f})$	$\begin{pmatrix} f & 0 \\ 1-f & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \tilde{f}-1 \\ 0 & \tilde{f} \end{pmatrix}$	-1	0	1

Benefits of this method

The Free Generators Theorem

- provides many choices of g_1 and g_2 over *any* characteristic
- offers a great amount of control of the degrees and form of the entries in our generators
- extends to an arbitrary number of generators

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Our approach provides

- a stronger version of the small modifications property
- a method to prevent against specific small relations
- precise conditions on our parameters for generating a large enough set of hash values

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The methods in our theorem

- can extend to the p-adic field $\mathrm{GL}_2(\mathbb{Q}_p)$
- would work for GL_n for other n
- yield the potential for a keyed hash function

Some Definitions

Note

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Absolute value

If $g_m \neq 0$, the **valuation** of g, $v(g)$, is m and the **absolute value** is $|g| = p^{-v(g)} = p^{-m}$.

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Distance

Let $[u], [v] \in \mathbb{P}^1$ be such that $[u] = [u_1 : u_2]$ and $[v] = [v_1 : v_2]$. Then the **distance** between $[u]$ and $[v]$ is

$$d([u],[v]) = \frac{||u \wedge v||}{||u||\,||v||} = \frac{|u_1 v_2 - u_2 v_1|}{\max\{|u_1|,|u_2|\}\max\{|v_1|,|v_2|\}}.$$

Neighbourhoods in $\mathbb{P}^1$

For $[u] \in \mathbb{P}^1$ we define $N\left([u], \frac{1}{p^{d+1}}\right)$ to be the closed neighbourhood of radius $\frac{1}{p^{d+1}}$.

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Proposition

For each $d \in \mathbb{N}_0$ there exist $p^d(p+1)$ disjoint neighbourhoods of radius $\frac{1}{p^{d+1}}$ such that for any point $[u] \in \mathbb{P}^1$, $N\left([u], \frac{1}{p^{d+1}}\right)$ is precisely one of these neighbourhoods. They are

1. for each $(a_0, a_1, ..., a_d) \in \mathbb{F}_p^{d+1}$, $\{[1 : a_0 + a_1 x + \ldots + a_d x^d + r] \mid r \in x^{d+1}\mathcal{O}\}$, and
2. for each $(0, a_1, ..., a_d) \in \mathbb{F}_p^{d+1}$, $\{[a_1 x + \ldots + a_d x^d + r : 1] \mid r \in x^{d+1}\mathcal{O}\}$.

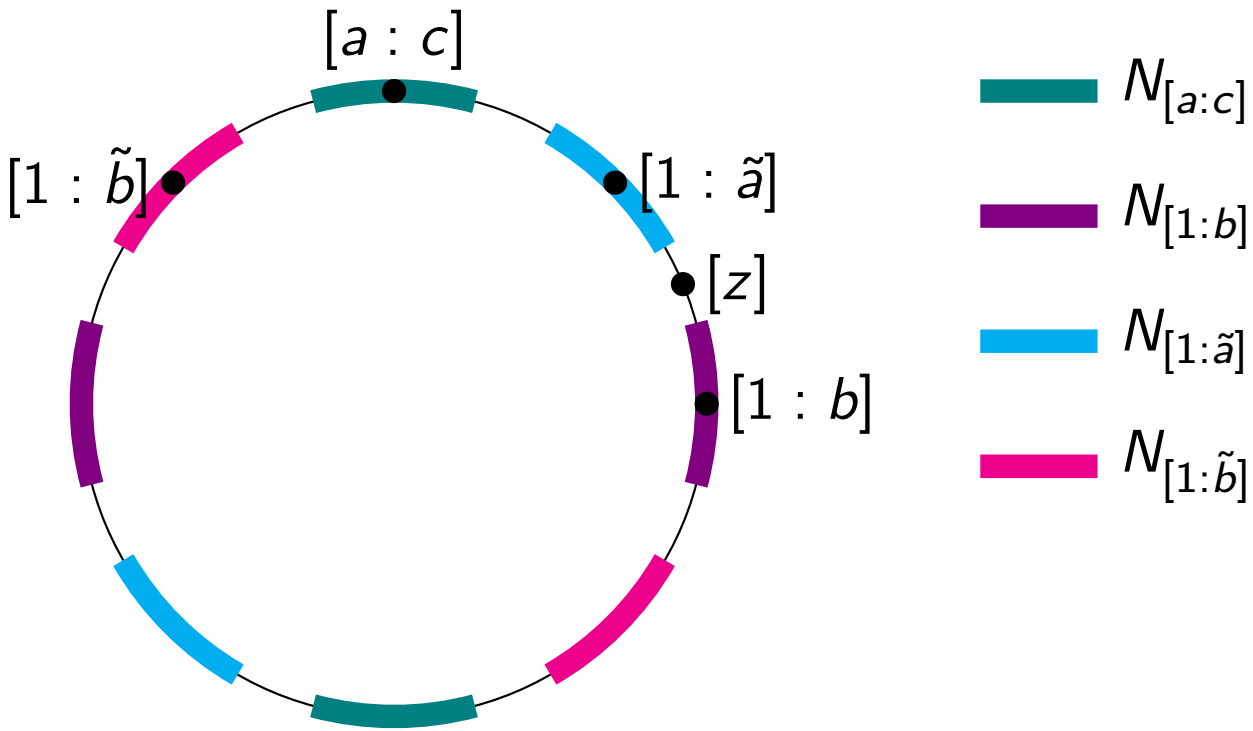


Figure: A visual representation of the $\frac{1}{p^{d+1}}$-neighbourhoods of the eigenvectors of A and B and the point $[z]$. Conditions Ξ_1 and Ξ_3 of Theorem 1 ensure these neigbourhoods are disjoint and the point $[z]$ must lie outside each neighbourhood.

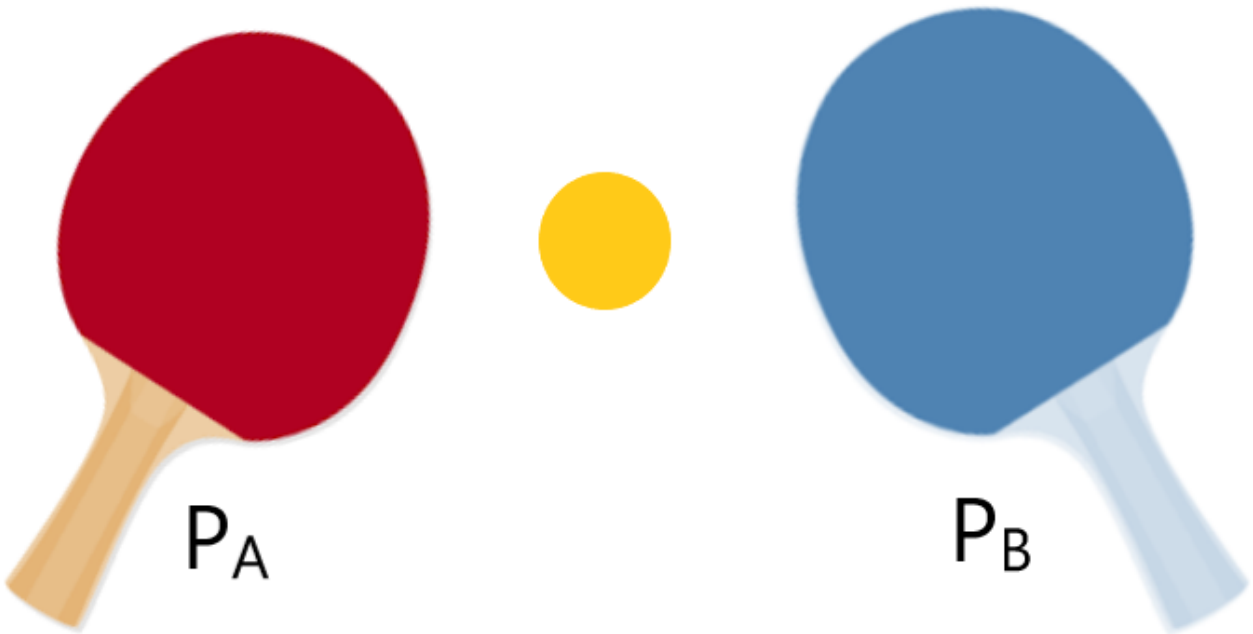
Ping-Pong Lemma (Jacques Tits, 1972)

Given two cyclic groups $\langle A \rangle$ and $\langle B \rangle$, acting on $\mathbb{P}^1$ with associated disjoint sets P_A and P_B in $\mathbb{P}^1$ with the property

- for any $g \in \langle A \rangle$, $g : \mathbb{P}^1 \setminus P_A \to P_A$ and
- for any $g \in \langle B \rangle$, $g : \mathbb{P}^1 \setminus P_B \to P_B$

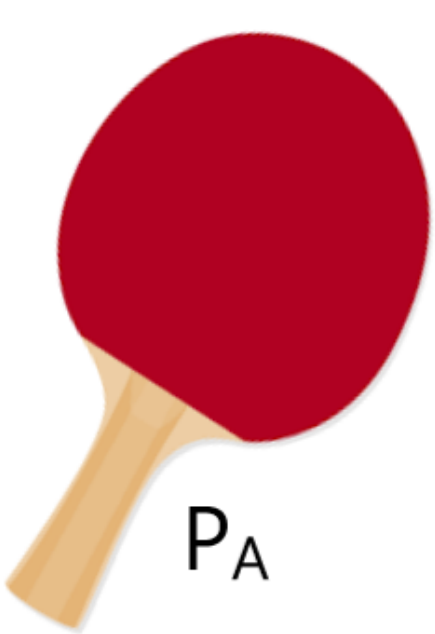
Then, any nontrivial word w in $\{A, B\}$ must necessarily map a point outside of $P_A \cup P_B$ to either P_A or P_B, and thus cannot be identity.

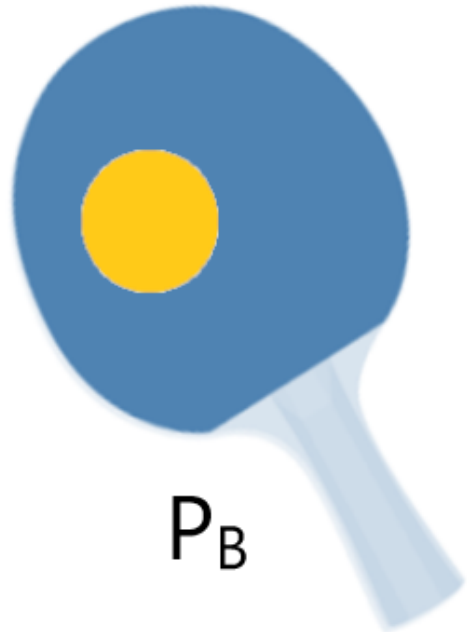
Consider the action of the word $A^2B^{-1}A^5B^4$ on $[z]$



$[z]$

B^4 maps $[z]$ to P_B





$$B^4 \cdot [z]$$

A^5 maps $B^4 \cdot [z]$ to P_A

$$A^5 B^4 \cdot [z]$$

B^{-1} maps $A^5B^4 \cdot [z]$ to P_B

$$B^{-1}A^5B^4 \cdot [z]$$

A^2 maps $B^{-1}A^5B^4 \cdot [z]$ to P_B

$$A^2B^{-1}A^5B^4 \cdot [z]$$

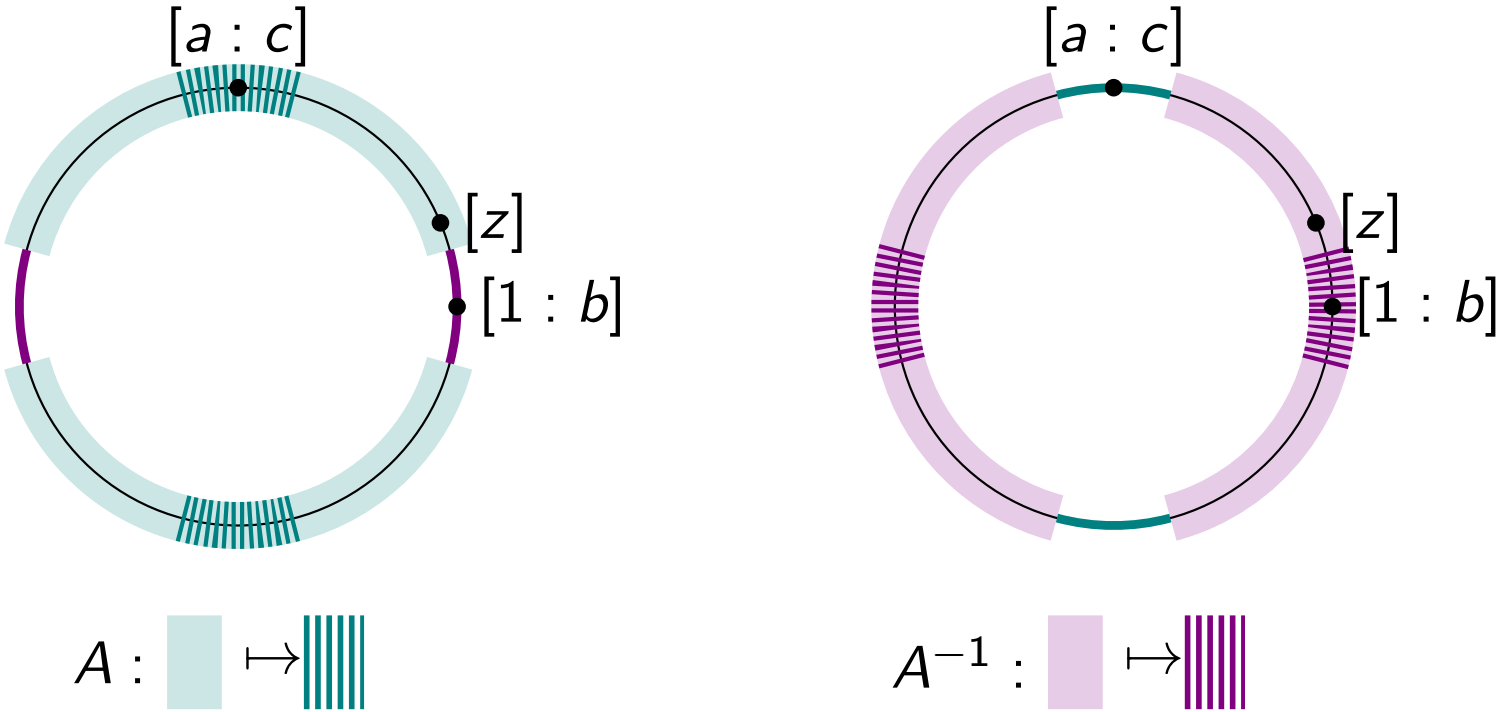
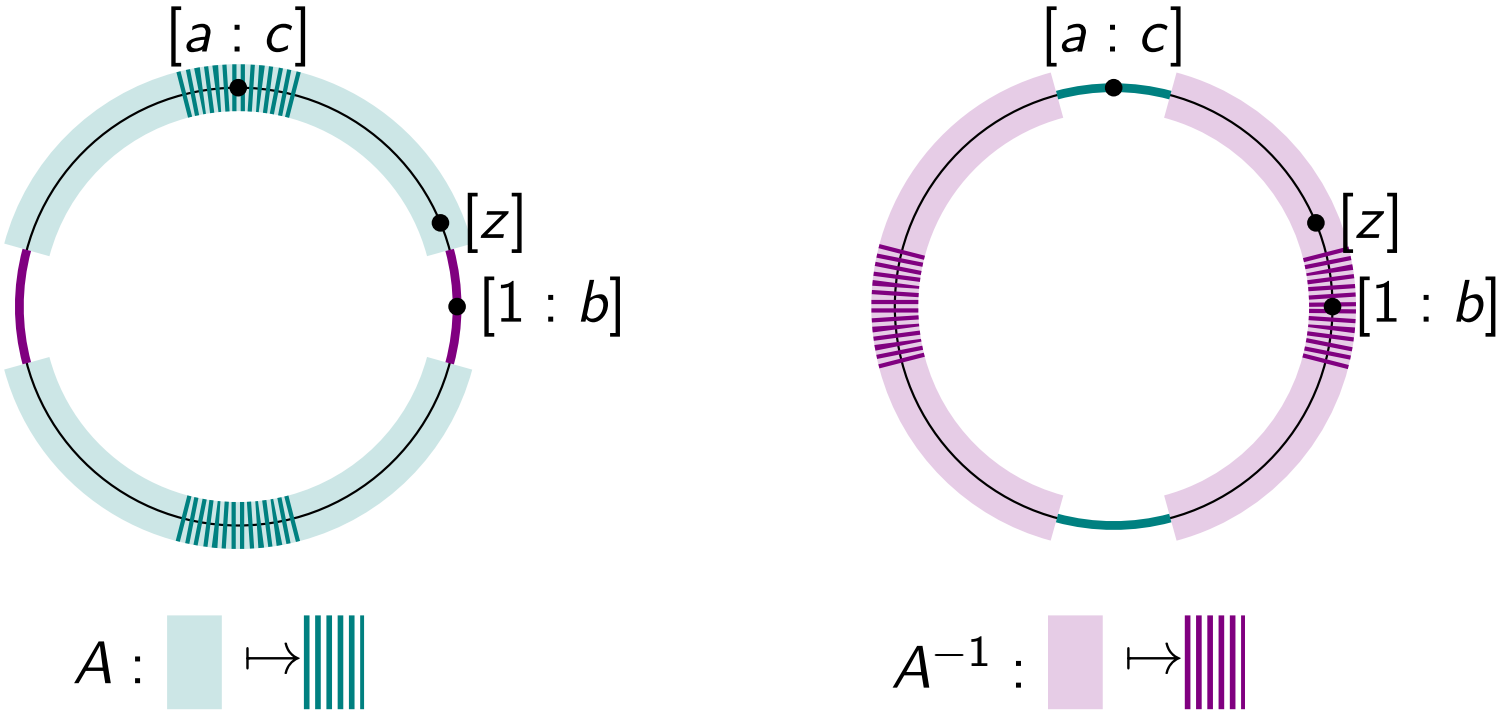


Figure: A visual representation of the action of A and the action of A^{-1} on $\mathbb{P}^1$. We see that A maps $\mathbb{P}^1 \setminus N_{[1:b]}$ to $N_{[a:c]}$ and A^{-1} maps $\mathbb{P}^1 \setminus N_{[a:c]}$ to $N_{[1:b]}$.



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We choose $P_A = N_{[a:c]} \cup N_{[1:b]}$ and $P_B = N_{[1:\tilde{a}]} \cup N_{[1:\tilde{b}]}$ and let $[z]$ be a point outside of P_A or P_B. This gives that $\langle A \rangle$ maps $\mathbb{P}^1 \setminus P_A$ to P_A.

Final Remarks

- potential issues from the introduction of the determinant are fixed by padding or choosing $\det(A) = \det(B)$
- our hash function constructions are resistant to attacks on the Zémor-Tillich hash function

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Thank you!

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