

Complexity Bound on Semaev's Naive Index Calculus Method for ECDLP

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 - The dominant part is **solving the Point Decomposition Problem**, to which Gröbner basis computation is applied.
 - To understand the **computational behavior** of Gröbner basis computation very precisely, we focus on **S-polynomial** computation, the main step of Gröbner basis computation.
 - Introducing the notion **signature**, we find that the **generated sequence of S-polynomials** is very similar to **regular polynomial remainder sequence** in the Euclidean algorithm for GCD computation.

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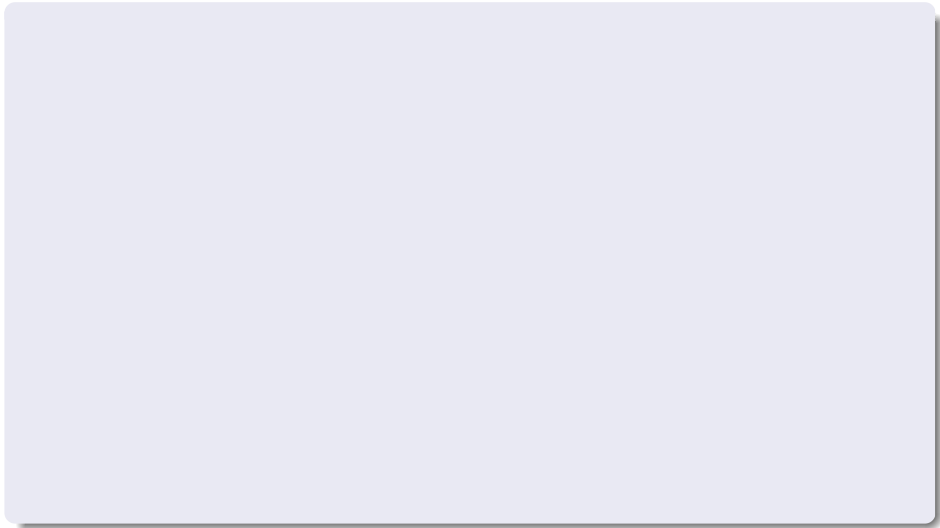
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- There are a number of advanced (improved) ICM methods for **ECDLP**. Especially, ICMs with the **Weil descent technique** are studied as the fastest ones. Also, there are several **conjectures** about upper bounds on the complexity of Gröbner basis computation used in ICMs.

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can be reduced to finding a solution $\alpha_1, \dots, \alpha_m \in V = \{x(P) \mid P \in \mathcal{B}\}$ such that

$$S_{m+1}(\alpha_1, \dots, \alpha_m, x(aP + bQ)) = 0,$$

for the **Semaev's summation polynomial** S_{m+1} .

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the problem is reduced to solving the following system:

$$\text{Sys}(\beta) : \begin{cases} S_{m+1}(x_1, \dots, x_m, \beta) & = & 0 \\ F(x_1) & = & 0 \\ \vdots & \vdots & \vdots \\ F(x_m) & = & 0 \end{cases}$$

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Let r be a positive integer greater than 1. There is a polynomial $S_r(x_1, \dots, x_r)$ in r variables over K , called the (Semaev's) summation polynomial of order r , which satisfies the following:

For each $b_1, \dots, b_r \in \overline{K}$ with $S_r(b_1, \dots, b_r) = 0$, there are points P_1, \dots, P_r in $E(\overline{K})$ such that

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$$S_r = \text{Res}_y(S_{r-1}(x_1, \dots, x_{r-2}, y), S_3(x_{r-1}, x_r, y)),$$

where Res_y denotes the resultant with respect to y .

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- Let C_{dec} be the **average cost** for solving $Sys(x(aP + bQ))$.
- Then the total (average) cost is, at least,

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- In our setting, $\#V \approx t$ and $\mathcal{P}rob_{\text{PDP}} \approx \frac{t^m}{\gamma q}$, where $q = \#K$, that is, $K = \mathbb{F}_q$.

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if the lower bound holds for some values β and modulus q , it also holds for **almost every** β and q .

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- **Signature-Based Algorithm:** Signature-based algorithms, such as F_5 or its variants, are designed to discard **unnecessary S-polynomials** as much as possible, by using the notion **signature**. Since **reduction** operations are restricted to keep the signature, it is not proven rigidly that signature-based algorithms always produce lesser number of S-polynomials.
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Our approximation is **roughly examined** by computational experience.

Computational Experiments

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We set finite prime fields \mathbb{F}_q with $q = 2^B + c$, ($2^B \gg c$) such that \mathbb{F}_p^\times has a subgroup of order d . (Thus $F(x) = x^d - 1$ and $\delta = \deg_{x_i} S_{m+1} = 2^{m-2}$.)

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- $[**]$ denotes the average of **, and *Ratio* denotes the ratio of $[\#\mathcal{G}^*]$ and $\#\overline{NS}$, *PA* denotes the total number of multiplications of some polynomials and monomials.

Computational Experiments: $m = 4$ and $\delta = 8$

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Trivial-Ideal Case:

B	10	11	12	13	14	15
d	5	7	8	8	11	12
d^m	625	2401	4096	4096	14641	20736
$\#NS$	625	2401	4096	4096	14641	20736
$\#\overline{NS}$	625	2401	4096	4096	14560	20480
$[\#\mathcal{G}^*]$	625	2401	4096	4096	14301	19770
<i>Ratio</i>	1	1	1	1	0.98	0.97
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Non-Trivial-Ideal Case: $V(I)$ denotes the number of zeros.

B	12	13	15
d	5	6	7
d^m	625	1296	2401
$\#NS$	613.6	1287.8	2382.8
$\#\overline{NS}$	613.6	1287.8	2382.8
$\#[\mathcal{G}^*]$	613.6	1287.8	2382.8
<i>Ratio</i>	1	1	1
$\#V(I)$	11.4	8.2	18.2
$[PA]$	121820	408965	1173939

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d	12	20	30
$\#NS$	1728	8000	27000
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md^{m-1}	432	1200	2700
$[\#\mathcal{G}^*]$	947	2599	5744
<i>Ratio</i>	0.78	0.67	0.61
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$$d^m - (d - \delta)^m + 1 > \#\mathcal{G}^* > md^{m-1}$$

Euclidean Algorithm and Gröbner Basis Computation

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$$f_2 = f_0 - (a_1x + b_1)f_1$$

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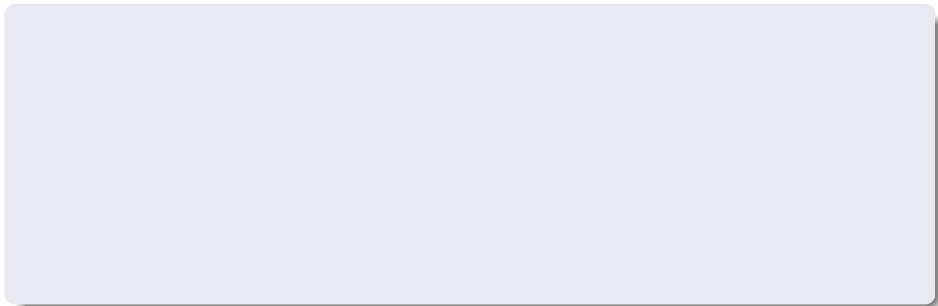
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In Buchberger algorithm, we have to compute the S-polynomial for every distinct pair (f_i, f_j) . Introducing **signature**, we may discard **unnecessary S-polynomials**.

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If f is given as $f = f(x, \beta)$, the condition is an algebraic condition on β .

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For each f_i in the PRS, A_i is a polynomial with the smallest degree and $\text{sig}(f_i) = x^{i-2}$.

Then, the signature of the S-polynomial $\text{Spol}(f_i, f_{i+1})$ coincides with that of $a_{i+1}xf_{i+1}$ and hence,

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Signature is a useful tool for analysing S-polynomials.

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By this condition, we can check signatures for which S-polynomials appear.

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As NS is a finite set, this relates to **Subresultant Theory**.

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For some value $\beta(= x(aP + bQ))$ and modulus p , if the ideal is \overline{NS} -semi-regular, then it is so for almost every value and modulus.

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